

# $\hbar$ -(Yangian) Deformation of Miura Map and Virasoro Algebra

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## Abstract

An  $\hbar$ -deformed Virasoro Poisson algebra is obtained using the Wakimoto realization of the Sugawara operator for the Yangian double  $DY_{\hbar}(sl_2)_c$  at the critical level  $c = -2$ .

## 1 Introduction

In this work we construct the Yangian deformed Miura map and the corresponding (deformed) Virasoro algebra. Since Drinfeld proposed in his works [8, 9], the Yangian algebra  $Y_{\hbar}(g)$  is known as a kind of Hopf algebra associated with rational solutions of quantum Yang-Baxter equation characterized by an additive parameter  $\hbar$ , and is a deformation of the universal enveloping algebra of the loop algebra  $g[u, u^{-1}]$  associated with the simple Lie algebra  $g$ . Since the rational solution of the Yang-Baxter equation is characterized by an additive parameter  $\hbar$ , we call the corresponding deformed algebras  $\hbar$ -deformations. Many faces of the Yangian algebra have been thoroughly studied in the literatures with emphasis in both algebraic aspects and physical applications. Algebraically, the development of studies of Yangian algebra is almost parallel to the development of studies of the so-called quantum algebra associated with trigonometric solutions of Yang-Baxter solution and characterized by a multiplicative parameter  $q$  (and hence the name  $q$ -deformed algebras [10]), and such parallelism became even more perfect after the work of Khoroshkin *et al* who successfully constructed the quantum double for Yangian algebra (the Yangian double [18]) and made a central extension for Yangian double [19, 20, 21, 17]. Physically the Yangian algebra (actually its quantum double  $DY_{\hbar}(g)$ ) is found to be the dynamical symmetry algebra of many massive quantum integrable field theories, among which we would like to mention the Thirring model, principal chiral model and nonlinear sigma model [30, 6, 23]. Generally speaking the field theoretical models yielding Yangian symmetries often correspond to certain scaling limits of some lattice statistical models away from criticality. In this sense the

Yangian algebra is in a position somewhere in between the usual Kac-Moody Lie algebra and the quantum algebra.

In the study of  $q$ -deformed Lie algebras there were a long standing problem which is not resolved until recently, i.e. the construction of a  $q$ -deformation of Virasoro algebra. After several attempts by different authors (see, e.g. [26]), in Ref.[15], Frenkel and Reshetikhin obtained a version of  $q$ -deformed Virasoro and together W algebras as  $q$ -deformed Gelfand-Dickey Poisson algebras. Later on, the quantum version of their algebras has also been obtained by several groups, see Refs. [28, 24, 2, 3, 13]. Vertex operators connected to such algebras have also been studied in Refs. [25, 29, 1]. The central idea of Ref. [15] can be briefly summarized as follows. In the undeformed case, the (quantum) Virasoro algebra can be constructed from the Kac-Moody algebra, e.g.  $\widehat{sl}_2$ , by means of Sugawara construction, and by an appropriate renormalization, the Virasoro generating function becomes a center in the formal completion of the universal enveloping algebra of  $\widehat{sl}_2$  at the critical level  $k = -2$  (the dual Coxeter number with a minus sign). The Poisson brackets for the Virasoro algebra can then be obtained from the Wakimoto realization of the  $\widehat{sl}_2$  Kac-Moody current in the limit  $k + 2 \rightarrow 0$ , and this construction has a natural connection to the famous Miura transformation (actually the Virasoro Poisson brackets has to be obtained via this transformation). In the  $q$ -deformed case, Frenkel and Reshetikhin [15] successfully made a parallel development. Using the Ding-Frenkel equivalence [7] of Drinfeld currents [11] and the Reshetikhin-Semenov-Tian-Shansky realization [27] of  $q$ -affine algebras they obtained the center of the formal completion of the  $q$ -algebra  $U_q(\widehat{sl}_2)_k$ . Then Using the  $q$ -deformed Wakimoto realization [4] of the Drinfeld currents they showed that that center is nothing but a  $q$ -deformation of the Miura map. Finally they obtained the Poisson bracket algebra for  $q$ -deformed Virasoro algebra using the  $q$ -Miura map at the critical level  $k = -2$ . The  $q$ -deformed W algebras are also obtained in a similar spirits.

It is interesting to ask that whether the constructions worked in undeformed and  $q$ -deformed cases also works in  $\hbar$ -deformed case. The answer is true but the construction is in some sense not straightforward, as will be shown in the main context of this paper. The  $\hbar$ -deformation of Virasoro algebra is our central object, and we feel that this algebra is important to be studied in detail because it is the corner stone of several important algebraic objects: (i) it is a deformation of the conventional Virasoro algebra; (ii) it is the scaling limit of the  $q$ -deformed Virasoro algebra obtained in Ref. [15]; (iii) its connection with Yangian algebra with center is precisely the sort of connections between the  $q$ -deformed Virasoro and  $q$ -affine algebras; (iv) it is the classical counterpart of the quantum  $\hbar$ -deformed Virasoro algebra obtained from the quantum  $q$ -Virasoro algebra by taking the scaling limit [16]. Moreover, the connections to the  $\hbar$ -deformed Miura map is also an important problem because that, while extended to algebras of higher rank, this may reveal a new kind of deformed Gelfand-Dickey equation and may also play some role in an  $\hbar$ -deformed Drinfeld-Sokolov reduction scheme [12].

The outline of this work is as follows. In Section 2 we shall collect necessary backgrounds

and formulas by a brief review of the Yangian doubles  $DY(gl_2)_c$  and  $DY_{\hbar}(sl_2)_c$ . Section 3 is devoted to the construction of  $\hbar$ -deformed Sugawara operator. Then, using the Wakimoto realization given in [22], we derive the  $\hbar$ -deformed Miura map in Section 4. In Section 5 we present the Poisson bracket for the  $\hbar$ -deformed Virasoro algebra and Section 6 is devoted to some discussions and out-lookings.

## 2 The Yangian algebras $DY_{\hbar}(gl_2)_c$ and $DY_{\hbar}(sl_2)_c$

The Yangian algebra we shall make use of is actually the central extension  $DY_{\hbar}(gl_2)_c$  and  $DY_{\hbar}(sl_2)_c$  of the quantum doubles of  $Y_{\hbar}(gl_2)$  and  $Y_{\hbar}(sl_2)$  respectively with central element  $c$ . These algebras can be realized in three equivalent ways, namely using the Chevalley generators, Drinfeld currents and Reshetkhin-Semenov-Tian-Shansky formalism. In this section we shall collect necessary formulas by making a brief review of the algebra  $DY_{\hbar}(gl_2)_c$  and treating the algebra  $DY_{\hbar}(sl_2)_c$  as the subalgebra of  $DY_{\hbar}(sl_2)_c$  modulo a Heisenberg subalgebra.

In terms of Drinfeld currents, the algebra  $DY_{\hbar}(gl_2)_c$  can be regarded as the formal completion of the algebra generated by the currents  $k_i^{\pm}(u)$  ( $i = 1, 2$ ),  $e^{\pm}(u)$ ,  $f^{\pm}(u)$  together with a derivative  $d$  and a center element  $c$  with the following generating relations [17],

$$\begin{aligned}
[d, e(u)] &= \frac{d}{du} e(u), \\
[d, f(u)] &= \frac{d}{du} f(u), \\
[d, k_i^{\pm}(u)] &= \frac{d}{du} k_i^{\pm}(u), \quad i = 1, 2, \\
k_i^{\pm}(u) k_j^{\pm}(v) &= k_j^{\pm}(v) k_i^{\pm}(u), \quad i, j = 1, 2, \\
\rho(u_- - v_+) k_i^+(u) k_i^-(v) &= k_i^-(v) k_i^+(u) \rho(u_+ - v_-), \quad i = 1, 2, \\
\rho(u_+ - v_- - \hbar) k_2^+(u) k_1^-(v) &= k_1^-(v) k_2^+(u) \rho(u_- - v_+ - \hbar), \\
\rho(u_+ - v_- + \hbar) k_1^+(u) k_2^-(v) &= k_2^-(v) k_1^+(u) \rho(u_- - v_+ + \hbar), \\
e(u) e(v) &= \frac{u - v + \hbar}{u - v - \hbar} e(v) e(u), \\
f(u) f(v) &= \frac{u - v - \hbar}{u - v + \hbar} f(v) f(u), \\
k_1^{\pm}(u) e(v) &= \frac{u_{\pm} - v}{u_{\pm} - v + \hbar} e(v) k_1^{\pm}(u), \\
k_2^{\pm}(u) e(v) &= \frac{u_{\pm} - v}{u_{\pm} - v - \hbar} e(v) k_2^{\pm}(u), \\
k_1^{\pm}(u) f(v) &= \frac{u_{\mp} - v + \hbar}{u_{\mp} - v} f(v) k_1^{\pm}(u), \\
k_2^{\pm}(u) f(v) &= \frac{u_{\mp} - v - \hbar}{u_{\mp} - v} f(v) k_2^{\pm}(u),
\end{aligned} \tag{1}$$

$$[e(u), f(v)] = \frac{1}{h} \left( \delta(u_- - v_+) k_2^+(u_-) k_1^+(u_-)^{-1} - \delta(u_+ - v_-) k_2^-(v_-) k_1^-(v_-)^{-1} \right),$$

where

$$\delta(u - v) = \sum_{n+m=-1} u^n v^m, \quad \delta(u - v)g(u) = \delta(u - v)g(v),$$

$$u_{\pm} = u \pm \frac{1}{4}\hbar c,$$

and the function  $\rho(u)$  is to be specified in the due course. The equivalence of the Drinfeld currents to the Chevalley generators is manifest in the following Laurent mode expansions of the currents,

$$e^{\pm}(u) = \pm \sum_{\substack{l \geq 0 \\ l < 0}} e[l] u^{-l-1}, \quad f^{\pm}(u) = \pm \sum_{\substack{l \geq 0 \\ l < 0}} f[l] u^{-l-1}, \quad k_i^{\pm}(u) = 1 \pm \hbar \sum_{\substack{l \geq 0 \\ l < 0}} k_i[l] u^{-l-1}.$$

In the main context of this paper, we shall actually need the equivalence between the Drinfeld currents and Reshetikhin-Semenov-Tian-Shansky realization, the latter is given by the following Yang-Baxter type relations [27],

$$\begin{aligned} R^{\pm}(u - v) L_1^{\pm}(u) L_2^{\pm}(v) &= L_2^{\pm}(v) L_1^{\pm}(u) R^{\pm}(u - v), \\ R^+(u_- - v_+) L_1^+(u) L_2^-(v) &= L_2^-(v) L_1^+(u) R^+(u_+ - v_-), \end{aligned} \quad (2)$$

where

$$R^{\pm}(u) = \rho^{\pm}(u) \begin{pmatrix} 1 & & & \\ & \frac{u}{u+\hbar} & \frac{\hbar}{u+\hbar} & \\ & \frac{\hbar}{u+\hbar} & \frac{u}{u+\hbar} & \\ & & & 1 \end{pmatrix},$$

$$\rho^{\pm}(u) = \left( \frac{\Gamma^2(\frac{1}{2} \mp \frac{u}{2\hbar})}{\Gamma(\mp \frac{u}{2\hbar}) \Gamma(1 \mp \frac{u}{2\hbar})} \right)^{\pm 1},$$

and the function  $\rho(u)$  appeared in (1) is precisely  $\rho^+(u)$ . Notice that the scalar functions  $\rho^{\pm}(u)$  in the  $R$  matrices  $R^{\pm}(u)$  are chosen such that the unitarity and crossing symmetry for the  $R$  matrices hold, i.e.,

$$R^+(u) R^-(-u) = 1, \quad (C \otimes id) R^{\pm}(u) (C \otimes id)^{-1} = R^{\mp}(-u - \hbar)^{t_1},$$

where  $t_1$  refers to the transpose in the first component space,  $C$  is the charge conjugation given in matrix form in the following,

$$C = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

The equivalence between the two realizations (1) and (2) is an analog of the well known Ding-Frenkel equivalence [7] of two similar realizations of  $q$ -affine algebras, see also [20] in the case of Yangian double  $DY_{\hbar}(sl_2)_c$ . In our case, the key point is that, in eq.(2), the  $L^{\pm}(u)$  can be given a Gauss decomposition,

$$L^{\pm}(u) = \begin{pmatrix} 1 & 0 \\ \hbar f^{\pm}(u_{\mp}) & 1 \end{pmatrix} \begin{pmatrix} k_1^{\pm}(u) & 0 \\ 0 & k_2^{\pm}(u) \end{pmatrix} \begin{pmatrix} 1 & \hbar e^{\pm}(u_{\pm}) \\ 0 & 1 \end{pmatrix}, \quad (3)$$

where the diagonal entries  $k_i^{\pm}(u)$  are identified with the Drinfeld currents  $k_i^{\pm}(u)$  in (1), and the off-diagonal entries  $e^{\pm}(u)$  and  $f^{\pm}(u)$  are related to the Drinfeld currents  $e(u)$  and  $f(u)$  by

$$e(u) = e^{+}(u) - e^{-}(u), \quad f(u) = f^{+}(u) - f^{-}(u).$$

The algebra  $DY_{\hbar}(gl_2)_c$  can be splitted into two subalgebras: the Yangian double  $DY_{\hbar}(sl_2)_c$  and a Heisenberg subalgebra. The Heisenberg subalgebra is generated by the currents

$$K^{\pm}(u) \equiv k_2^{\pm}(u + \hbar)k_1^{\pm}(u) - 1. \quad (4)$$

It is an easy practice to show that  $K^{\pm}(u)$  actually commute with all generating functions of  $DY_{\hbar}(gl_2)_c$  and thus generate a central subalgebra. The Yangian double  $DY_{\hbar}(sl_2)_c$  is thus obtained from  $DY_{\hbar}(gl_2)_c$  by taking the quotient with respect to this center. The resulting generating relations differ from that of  $DY_{\hbar}(gl_2)_c$  only in those involving the currents  $k_i^{\pm}(u)$ ,

$$\begin{aligned} [d, h^{\pm}(u)] &= \frac{d}{du} h^{\pm}(u), \\ [h^{\pm}(u), h^{\pm}(v)] &= 0, \\ h^{\pm}(u)e(v) &= \frac{u_{\pm} - v + \hbar}{u_{\pm} - v - \hbar} e(v)h^{\pm}(u), \\ h^{\pm}(u)f(v) &= \frac{u_{\mp} - v - \hbar}{u_{\mp} - v + \hbar} f(v)h^{\pm}(u), \\ h^{+}(u)h^{-}(v) &= \frac{u_{+} - v_{-} + \hbar}{u_{-} - v_{+} + \hbar} \cdot \frac{u_{-} - v_{+} - \hbar}{u_{+} - v_{-} - \hbar} h^{-}(v)h^{+}(u), \end{aligned}$$

where  $h^{\pm}(u)$  is defined as

$$h^{\pm}(u) = k_2^{\pm}(u)k_1^{\pm}(u)^{-1}. \quad (5)$$

In ending this section let us remark that one can recover the original currents  $k_{1,2}^{\pm}(u)$  from eq.(4) and (5) in the following form,

$$\begin{aligned} k_1^{+}(u) &= \prod_{l \geq 0} \frac{h^{+}(u - (2l + 1)\hbar)}{h^{+}(u - 2l\hbar)}, & k_2^{+}(u) &= \prod_{l \geq 0} \frac{h^{+}(u - (2l + 1)\hbar)}{h^{+}(u - (2l + 2)\hbar)}, \\ k_1^{-}(u) &= \prod_{l \geq 0} \frac{h^{-}(u + (2l + 2)\hbar)}{h^{-}(u + (2l + 1)\hbar)}, & k_2^{-}(u) &= \prod_{l \geq 0} \frac{h^{-}(u + 2l\hbar)}{h^{-}(u + (2l + 1)\hbar)}. \end{aligned} \quad (6)$$

These formulas will be used in Section 3.

### 3 $\hbar$ -deformed Sugawara construction

Let

$$L(u) = L^-(u - \frac{\hbar}{2})L^+(u + \frac{\hbar}{2})^{-1}. \quad (7)$$

The trace

$$l(u) = \text{tr}L(u) = L_{11}(u) + L_{22}(u),$$

as formal power series, would then lie in the formal completion of  $DY_{\hbar}(gl_2)_c$ . Following [27] we may conclude that at  $c = -2$  the coefficients of  $l(u)$  are central elements of the formal completion of  $DY_{\hbar}(gl_2)_c$ .

To express  $l(u)$  in terms of the equivalence between the Drinfeld currents and the Reshetikhin-Semenov-Tian-Shansky formalism, we will now set  $c = -2$ . It follows from (7) and (3) that

$$\begin{aligned} L_{11}(u) &= k_1^-(u - \frac{\hbar}{2})k_1^+(u + \frac{\hbar}{2})^{-1} + \hbar^2 k_1^-(u - \frac{\hbar}{2})e^+(u)k_2^+(u + \frac{\hbar}{2})^{-1}f^+(u + \hbar) \\ &\quad - \hbar^2 k_1^-(u - \frac{\hbar}{2})e^-(u)k_2^+(u + \frac{\hbar}{2})^{-1}f^+(u + \hbar), \\ L_{22}(u) &= k_2^-(u - \frac{\hbar}{2})k_2^+(u + \frac{\hbar}{2})^{-1} - \hbar^2 f^-(u - \hbar)k_1^-(u - \frac{\hbar}{2})e^+(u)k_2^+(u + \frac{\hbar}{2})^{-1} \\ &\quad + \hbar^2 f^-(u - \hbar)k_1^-(u - \frac{\hbar}{2})e^-(u)k_2^+(u + \frac{\hbar}{2})^{-1}. \end{aligned} \quad (8)$$

The last two equations have not yet been written purely in terms of Drinfeld currents. In order to do so, we have to combine  $e^{\pm}(u)$  into  $e(u)$  and  $f^{\pm}(u)$  into  $f(u)$ . The first step will of course be moving the  $f^+(u + \hbar)$  from the right of  $k_2^+(u + \frac{\hbar}{2})^{-1}$  to the left in  $L_{11}(u)$ ,  $f^-(u - \hbar)$  from the left of  $k_1^-(u - \frac{\hbar}{2})^{-1}$  to the right in  $L_{22}(u)$ . To achieve this we have to use the commutation relations for  $f^{\pm}(u)$  and  $k_i^{\pm}(v)$ . The required relations read

$$\begin{aligned} f^-(v_+)k_1^-(u) &= \frac{u-v}{u-v+\hbar}k_1^-(u)f^-(v_+) + \frac{\hbar}{u-v+\hbar}f^-(u_+)k_1^-(u), \\ k_2^+(v)^{-1}f^+(u_-) &= \frac{u-v}{u-v+\hbar}f^+(u_-)k_2^+(v)^{-1} + \frac{\hbar}{u-v+\hbar}k_2^+(v)^{-1}f^+(v_-). \end{aligned}$$

Multiplying by  $u - v + \hbar$ , we can see that, at points  $v = u + \hbar$ , the last equations become

$$\begin{aligned} f^-(u_+)k_1^-(u) &= k_1^-(u)f^-(u_+ + \hbar), \\ k_2^+(u)^{-1}f^+(u_-) &= f^+(u_- - \hbar)k_2^+(u)^{-1}. \end{aligned}$$

Notice that, when  $c = -2$ , we have  $u_{\pm} = u \mp \frac{1}{2}\hbar$ . Therefore, the above equations can be written as

$$\begin{aligned} f^-(u - \hbar)k_1^-(u - \frac{1}{2}\hbar) &= k_1^-(u - \frac{1}{2}\hbar)f^-(u), \\ k_2^+(u + \frac{1}{2}\hbar)^{-1}f^+(u + \hbar) &= f^+(u)k_2^+(u + \frac{1}{2}\hbar)^{-1}. \end{aligned} \quad (9)$$

Substituting (9) into (8), we are led to

$$\begin{aligned} L_{11}(u) &= k_1^-(u - \frac{\hbar}{2})k_1^+(u + \frac{\hbar}{2})^{-1} + \hbar^2 k_1^-(u - \frac{\hbar}{2}) [e^+(u) - e^-(u)] f^+(u)k_2^+(u + \frac{\hbar}{2})^{-1}, \\ L_{22}(u) &= k_2^-(u - \frac{\hbar}{2})k_2^+(u + \frac{\hbar}{2})^{-1} - \hbar^2 k_1^-(u - \frac{\hbar}{2})f^-(u) [e^+(u) - e^-(u)] k_2^+(u + \frac{\hbar}{2})^{-1}. \end{aligned}$$

Finally, we have the expression for  $l(u)$ ,

$$l(u) = k_1^-(u - \frac{\hbar}{2})k_1^+(u + \frac{\hbar}{2})^{-1} + k_2^-(u - \frac{\hbar}{2})k_2^+(u + \frac{\hbar}{2})^{-1} + \hbar^2 k_1^-(u - \frac{\hbar}{2}) : e(u)f(u) : k_2^+(u + \frac{\hbar}{2})^{-1}, \quad (10)$$

where

$$: e(u)f(u) := e(u)f^+(u) - f^-(u)e(u). \quad (11)$$

The final equation (10) is just the required  $\hbar$ -deformed Sugawara operator.

## 4 Wakimoto module of $DY_{\hbar}(sl_2)_c$ and $\hbar$ -deformed Miura map

Given the  $\hbar$ -deformed Sugawara construction, our next goal is to show its connection to the corresponding (deformed) Miura map. This can be fulfilled by making use of the Wakimoto module of the Yangian double.

We shall adopt the Wakimoto module of  $DY_{\hbar}(sl_2)_c$  given by Konno [22]<sup>1</sup>.

Introduce the following three sets of Heisenberg algebras with generators  $\lambda_n$ ,  $b_n$ ,  $c_n$   $n \in \mathbb{Z} - \{0\}$ ,  $\exp(\pm q_\lambda)$ ,  $\exp(\pm q_b)$ ,  $\exp(\pm q_c)$ ,  $p_\lambda$ ,  $p_b$ , and  $p_c$ ,

$$\begin{aligned} [\lambda_m, \lambda_n] &= \frac{k+2}{2}m\delta_{m+n,0}, & [p_\lambda, q_\lambda] &= \frac{k+2}{2}, \\ [b_m, b_n] &= -m\delta_{m+n,0}, & [p_b, q_b] &= -1, \\ [c_m, c_n] &= m\delta_{m+n,0}, & [p_c, q_c] &= 1. \end{aligned} \quad (12)$$

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<sup>1</sup>Our notations here defers from that of Konno in [22] in the following way: the boson  $\lambda$  corresponds to  $a_\Phi$  of Konno, and  $b$  and  $c$  correspond to  $a_\phi$  and  $a_\chi$  respectively, and there is a shift of spectral parameters in the bosonization formulas of Drinfeld current because our starting definition of the Yangian double  $DY_{\hbar}(sl_2)_c$  defers from that of Konno by such a shift.

For  $X = \lambda, b, c$ , define

$$X(u; A, B) = \sum_{n>0} \frac{X_{-n}}{n} (u + A\hbar)^n - \sum_{n>0} \frac{X_n}{n} (u + B\hbar)^{-n} + \log(u + B\hbar)p_X + q_X,$$

and together,  $X(u; A) = X(u; A, A)$ . We also use the abbreviations

$$\begin{aligned} X^+(u; B) &= - \sum_{n>0} \frac{X_n}{n} (u + B\hbar)^{-n}, \\ X^-(u; A) &= \sum_{n>0} \frac{X_{-n}}{n} (u + A\hbar)^n. \end{aligned}$$

The triple-mode Fock space is defined as follows. Let  $|0\rangle$  be a vector satisfying

$$X_n|0\rangle = 0, \quad n > 0; \quad p_X|0\rangle = 0.$$

Then  $|l, s, t\rangle \equiv \exp\left(\frac{l}{k+2}q_\lambda + sq_b + tq_c\right)|0\rangle$  is a vacuum state with  $\lambda, b, c$  charges  $l, -s, t$  respectively. The Fock space is generated by the action of  $\lambda_{-n}, b_{-n}, c_{-n} (n > 0)$  on  $|l, s, t\rangle$ ,

$$\mathcal{F}_{l,s,t} = \left\{ \prod_{n>0} \lambda_{-n} \prod_{n'>0} b_{-n'} \prod_{n''>0} c_{-n''} \right\} |l, s, t\rangle.$$

On  $\mathcal{F}_{l,s,t}$ , the normal ordering of  $\exp(X(u; A, B))$  is defined as

$$: \exp(X(u; A, B)) := \exp(X^-(u; A)) \exp(q_X)(u + B\hbar)^{p_X} \exp(X^+(u; B)).$$

Now we are ready to define the Wakimoto module for the Yangian double  $DY_{\hbar}(sl_2)_c$ . This is nothing but a homomorphism from the above defined Heisenberg algebras to  $DY_{\hbar}(sl_2)_c$  under which the action of the Drinfeld currents on  $\mathcal{F}_{l,s,t}$  is given by [22]

$$\begin{aligned} c &= k, \\ d &= d_\lambda + d_b + d_c, \\ d_\lambda &= \frac{2}{k+2} \left( \lambda_{-1} p_\lambda + \sum_{n>0} \lambda_{-(n+1)} \lambda_n \right), \\ d_b &= -b_{-1} p_b - \sum_{n>0} b_{-(n+1)} b_n, \\ d_c &= c_{-1} p_c + \sum_{n>0} c_{-(n+1)} c_n, \\ h^+(u) &= \exp \left[ \lambda^+(u; -\frac{3}{4}k) - \lambda^+(u; -(\frac{3}{4}k + 2)) \right. \\ &\quad \left. + b^+(u; -\frac{3}{4}k) - b^+(u; -(\frac{3}{4}k + 2)) \right] \left( \frac{u - \frac{3}{4}k\hbar}{u - (\frac{3}{4}k + 2)\hbar} \right)^{p_\lambda + p_b}, \end{aligned}$$



$$\begin{aligned}
h^-(u) &= \exp \left[ \frac{2}{k+2} \left( \lambda^-(u; -(\frac{5}{4}k+3)) - \lambda^-(u; -(\frac{1}{4}k+1)) \right) \right] \\
&\quad \times \exp \left[ b^-(u; -(\frac{5}{4}k+3)) - b^-(u; -(\frac{1}{4}k+1)) \right], \\
e(u) &= -\frac{1}{\hbar} : (\exp(-c(u; -(k+1))) - \exp(-c(u; -(k+2)))) \\
&\quad \times \exp(-b(u; -(k+1), -(k+2))) :, \\
f(u) &= \frac{1}{\hbar} : \left( \exp \left[ \lambda^+(u; -\frac{1}{2}k) - \lambda^+(u; -(\frac{1}{2}k+2)) \right] \left( \frac{u - \frac{1}{2}k\hbar}{u - (\frac{1}{2}k+2)\hbar} \right)^{p_\lambda} \right. \\
&\quad \times \exp \left[ b(u; -(\frac{1}{2}k+1), -\frac{1}{2}k) + c(u; -(\frac{1}{2}k+1)) \right] \\
&\quad \left. - \exp \left[ \frac{2}{k+2} \left( \lambda^-(u; -(\frac{3}{2}k+3)) - \lambda^-(u; -(\frac{1}{2}k+1)) \right) \right] \right. \\
&\quad \left. \times \exp \left[ b(u; -(\frac{3}{2}k+3), -(\frac{3}{2}k+2)) + c(u; -(\frac{3}{2}k+2)) \right] \right) : .
\end{aligned}$$

Introducing the notations

$$\begin{aligned}
\Upsilon^+(u) &= (u - (\frac{1}{2}k+1)\hbar)^{p_\lambda} \exp \left( \lambda^+(u; -(\frac{1}{2}k+1)) \right), \\
\Upsilon^-(u) &= \exp \left( \frac{2}{k+2} \lambda^-(u; -(k+2)) \right),
\end{aligned}$$

the expressions for  $h^\pm(u)$  and  $f(u)$  can be recasted into a relatively shorter form,

$$\begin{aligned}
h^+(u) &= \Upsilon^+(u_- + \hbar) \Upsilon^+(u_- - \hbar)^{-1} \left( \frac{u - \frac{3}{4}k\hbar}{u - (\frac{3}{4}k+2)\hbar} \right)^{p_b} \\
&\quad \times \exp \left[ b^+(u; -\frac{3}{4}k) - b^+(u; -(\frac{3}{4}k+2)) \right], \tag{13}
\end{aligned}$$

$$\begin{aligned}
h^-(u) &= \Upsilon^-(u_+ - \frac{k+2}{2}\hbar) \Upsilon^-(u_+ + \frac{k+2}{2}\hbar)^{-1} \\
&\quad \times \exp \left[ b^-(u; -(\frac{5}{4}k+3)) - b^-(u; -(\frac{1}{4}k+1)) \right], \tag{14}
\end{aligned}$$

$$\begin{aligned}
f(u) &= \frac{1}{\hbar} : \Upsilon^+(u + \hbar) \Upsilon^+(u - \hbar)^{-1} \\
&\quad \times \exp \left[ b(u; -(\frac{1}{2}k+1), -\frac{1}{2}k) + c(u; -(\frac{1}{2}k+1)) \right] \\
&\quad - \Upsilon^-(u - \frac{k+2}{2}\hbar) \Upsilon^-(u + \frac{k+2}{2}\hbar)^{-1} \\
&\quad \times \exp \left[ b(u; -(\frac{3}{2}k+3), -(\frac{3}{2}k+2)) + c(u; -(\frac{3}{2}k+2)) \right] : .
\end{aligned}$$

In order to obtain the  $\hbar$ -deformed Miura map, we need to express  $l(u)$  in terms of Laurent modes of only one of the three bosons,  $\lambda$ . To achieve this goal, we need an bosonic expression for  $k_i^\pm(u)$ , which can be obtained by substituting (13) and (14) into (6),

$$\begin{aligned}
k_1^+(u) &= \frac{\Upsilon^+(u - \frac{k}{2}\hbar)}{\Upsilon^+(u - \frac{k}{2}\hbar + \hbar)} \left( \frac{u - (k+1)\hbar}{u - k\hbar} \right)^{p_b} \\
&\quad \times \exp(b^+(u; -(k+1)) - b^+(u; -k)), \\
k_1^-(u) &= \prod_{l \geq 0} \frac{\Upsilon^-(u + k\hbar + (2l+2)\hbar)}{\Upsilon^-(u + k\hbar + (2l+3)\hbar)} \frac{\Upsilon^-(u + (2l+1)\hbar)}{\Upsilon^-(u + 2l\hbar)} \\
&\quad \times \exp(b^-(u; -(k+1)) - b^-(u; -(k+2))), \\
k_2^+(u) &= \frac{\Upsilon^+(u - \frac{k}{2}\hbar)}{\Upsilon^+(u - \frac{k}{2}\hbar - \hbar)} \left( \frac{u - (k+1)\hbar}{u - (k+2)\hbar} \right)^{p_b} \\
&\quad \times \exp(b^+(u; -(k+1)) - b^+(u; -(k+2))), \\
k_2^-(u) &= \prod_{l \geq 0} \frac{\Upsilon^-(u + k\hbar + (2l+2)\hbar)}{\Upsilon^-(u + k\hbar + (2l+1)\hbar)} \frac{\Upsilon^-(u + (2l-1)\hbar)}{\Upsilon^-(u + 2l\hbar)} \\
&\quad \times \exp(b^-(u; -(k+3)) - b^-(u; -(k+2))). \tag{15}
\end{aligned}$$

On the other hand, on the Fock space  $\mathcal{F}_{l,s,t}$ , the normal ordering for  $e(u)$  and  $f(u)$  is given by

$$: e(u)f(u) := - \int_{C_1} dv \frac{e(u)f(v)}{u - v - \frac{k+2}{2}\hbar} + \int_{C_2} dv \frac{f(v)e(u)}{u - v + \frac{k+2}{2}\hbar}, \tag{16}$$

where  $C_1$  and  $C_2$  are respectively circles of radius  $|v| > |u - \frac{k+2}{2}\hbar|$  and  $|v| < |u + \frac{k+2}{2}\hbar|$ . Applying the concrete expressions for  $e(u)$  and  $f(v)$  in (16), we can get

$$\begin{aligned}
: e(u)f(u) &:= \frac{1}{\hbar^2} \left( - \frac{\Upsilon^+(u - (\frac{k}{2} - 1)\hbar)}{\Upsilon^+(u - (\frac{k}{2} + 1)\hbar)} \right. \\
&\quad \times : \exp(b(u; -(k+1), -k) - b(u; -(k+1), -(k+2))) : \\
&\quad + \frac{\Upsilon^+(u - (\frac{k+2}{2} - 1)\hbar)}{\Upsilon^+(u - (\frac{k+2}{2} + 1)\hbar)} \\
&\quad \times : \exp(b(u; -(k+2), -(k+1)) - b(u; -(k+1), -(k+2))) : \\
&\quad + \frac{\Upsilon^-(u)}{\Upsilon^-(u + (k+2)\hbar)} \\
&\quad \times : \exp(b(u; -(k+2), -(k+1)) - b(u; -(k+1), -(k+2))) : \\
&\quad - \frac{\Upsilon^-(u - \hbar)}{\Upsilon^-(u - (k+1)\hbar)} \\
&\quad \times : \exp(b(u; -(k+3), -(k+2)) - b(u; -(k+1), -(k+2))) : \tag{17}
\end{aligned}$$

Substituting equations (15)-(17) into (10) and after some algebra, we finally obtain

$$l(u) = \Lambda(u - \frac{\hbar}{2}) + \Lambda(u + \frac{\hbar}{2})^{-1}, \tag{18}$$

where

$$\Lambda(u) = \Lambda^+(u)\Lambda^-(u) \quad (19)$$

and

$$\begin{aligned} \Lambda^+(u) &= \frac{\Upsilon^+\left(u - \frac{k+2}{2}\hbar + \frac{\hbar}{2}\right)}{\Upsilon^+\left(u - \frac{k+2}{2}\hbar - \frac{\hbar}{2}\right)}, \\ \Lambda^-(u) &= \prod_{l \geq 0} \frac{\Upsilon^-\left(u + (k+2)\hbar + (2l+1)\hbar - \frac{\hbar}{2}\right)}{\Upsilon^-\left(u + (k+2)\hbar + (2l+1)\hbar + \frac{\hbar}{2}\right)} \frac{\Upsilon^-\left(u + (2l+1)\hbar + \frac{\hbar}{2}\right)}{\Upsilon^-\left(u + (2l+1)\hbar - \frac{\hbar}{2}\right)}. \end{aligned}$$

Notice that eq.(18) contains only expressions involving the Laurent modes of  $\lambda$  and thus provides a map from one free  $\hbar$ -deformed bosonic field  $\lambda$  to the  $\hbar$ -deformed Sugawara operator  $l(u)$ . This is precisely the desired  $\hbar$ -deformed Miura map.

We have to remark that, though in the form of eq.(18) the  $\hbar$ -deformed Miura map looks very similar to the  $q$ -deformed version, the actual way of mapping from the bosonic field  $\lambda$  to  $l(u)$  is much more complicated than the  $q$ -deformed case. The complexity comes about in the infinite product structure in the expression for  $\Lambda^-(u)$ . However, despite such complexities the operator product between  $\Lambda^+$  and  $\Lambda^-$  is rather simple. It reads

$$\Lambda^+(u)\Lambda^-(v) = \frac{\rho(u-v-(k+2)\hbar)}{\rho(u-v)}\Lambda^-(v)\Lambda^+(u).$$

Another remark is concerned with the fact that the Sugawara operator  $l(u)$  is associated with the two-dimensional representation of  $sl_2$ . To obtain analogous operators associated with higher dimensional representations of  $sl_2$ , one can apply the fusing procedure which is similar to the  $q$ -deformed case, i.e. the operator  $l^{(n)}(u)$  associated with the  $(n+1)$ -dimensional representation of  $sl_2$  can be obtained from the following iterative relation,

$$\begin{aligned} l^{(1)}(u - n\hbar)l^{(n)}(u) &= l^{(n+1)}(u) + l^{(n-1)}(u), \\ l^{(1)}(u) &= l(u), \quad l^{(0)}(u) = 1. \end{aligned}$$

The explicit form for  $l^{(n)}(u)$  reads

$$\begin{aligned} l^{(n)}(u) &= \Lambda\left(u - \frac{\hbar}{2}\right)\Lambda\left(u - \frac{3\hbar}{2}\right)\Lambda\left(u - \frac{5\hbar}{2}\right)\dots\Lambda\left(u - \frac{(2n-1)\hbar}{2}\right) \\ &+ \Lambda\left(u + \frac{\hbar}{2}\right)^{-1}\Lambda\left(u - \frac{3\hbar}{2}\right)\Lambda\left(u - \frac{5\hbar}{2}\right)\dots\Lambda\left(u - \frac{(2n-1)\hbar}{2}\right) \\ &+ \Lambda\left(u + \frac{\hbar}{2}\right)^{-1}\Lambda\left(u - \frac{\hbar}{2}\right)^{-1}\Lambda\left(u - \frac{5\hbar}{2}\right)\dots\Lambda\left(u - \frac{(2n-1)\hbar}{2}\right) \\ &+ \Lambda\left(u + \frac{\hbar}{2}\right)^{-1}\Lambda\left(u - \frac{\hbar}{2}\right)^{-1}\Lambda\left(u - \frac{3\hbar}{2}\right)^{-1}\dots\Lambda\left(u - \frac{(2n-1)\hbar}{2}\right) \\ &+ \dots\dots \\ &+ \Lambda\left(u + \frac{\hbar}{2}\right)^{-1}\Lambda\left(u - \frac{\hbar}{2}\right)^{-1}\Lambda\left(u - \frac{3\hbar}{2}\right)^{-1}\dots\Lambda\left(u - \frac{(2n-3)\hbar}{2}\right)^{-1}. \end{aligned}$$

## 5 Poisson brackets for the $\hbar$ -deformed Virasoro algebra

Let us recall that in the undeformed case, the Miura map provides a free field representation of the classical (i.e. Poisson bracket) Virasoro algebra. In the  $q$ -deformed case, such a map also gives rise from the Poisson brackets for a  $q$ -deformed bosonic field to a  $q$ -deformed Virasoro Poisson algebra. In this section we shall show how we can obtain an analogous  $\hbar$ -deformed algebra.

First let us explain how a Poisson brackets could arise from a purely quantum theory. The key point is as follows. When  $k+2$  approaches zero, the commutation relations in (12), divided by  $k+2$ , naturally induces a Poisson bracket structure,

$$\{\lambda_m, \lambda_n\} = \frac{1}{2}m\delta_{m+n,0}, \quad \{p_\lambda, q_\lambda\} = \frac{1}{2}. \quad (20)$$

This Poisson structure turn the quantum  $\hbar$ -deformed free field  $\lambda(u; A, B)$  into a classical object. In the meantime, all functions of the quantum field  $\lambda$  are also turned into classical ones, i.e. the noncommuting objects become now commutative and the original commutation relations are turned into Poisson brackets.

To obtain the Poisson bracket for the  $\hbar$ -deformed Virasoro algebra, we need to take the limit of  $l(u)$  as  $k+2 \rightarrow 0$ . Simply substituting  $k+2=0$  into the expressions of  $\Upsilon^\pm(u)$  does not make sense because  $\Upsilon^-(u)$  is not a well defined object as  $k+2 \rightarrow 0$ . However, at level of  $\Lambda^\pm(u)$ , well defined limits could be obtained. The limits of  $\Lambda^\pm(u)$  read

$$\begin{aligned} \Lambda^\pm(u) &= A^\pm(u - \frac{1}{2}\hbar)A^\pm(u + \frac{1}{2}\hbar)^{-1}, \\ A^+(u) &= u^{-p_\lambda} \exp \left\{ \sum_{l>0} \frac{\lambda_l}{l} u^{-l} \right\}, \\ A^-(u) &= \prod_{l \geq 0} B^-(u + (2l+1)\hbar), \\ B^-(u) &= \exp \left\{ \sum_{n>0} 2\hbar \lambda_{-n} u^{n-1} \right\}. \end{aligned}$$

Using the Poisson brackets (20), one can easily calculates

$$\begin{aligned} \{A^+(u), B^-(v)\} &= \hbar \frac{\partial}{\partial v} \sum_{n>0} \frac{1}{n} \left(\frac{v}{u}\right)^n A^+(u) B^-(v) \\ &= -\hbar \frac{\partial}{\partial v} \log \left(1 - \frac{v}{u}\right) A^+(u) B^-(v), \quad (|u| > |v|) \\ \{B^-(u), A^+(v)\} &= \hbar \frac{\partial}{\partial u} \log \left(1 - \frac{u}{v}\right) B^-(u) A^+(v), \quad (|v| > |u|) \end{aligned}$$

from which we obtain

$$\begin{aligned}
\{A^+(u), A^-(v)\} &= -\hbar \frac{\partial}{\partial v} \sum_{l \geq 0} \log \left( 1 - \frac{v + (2l+1)\hbar}{u} \right) A^+(u) A^-(v) \\
&= -\hbar \frac{\partial}{\partial v} \log \left[ \prod_{l \geq 0} \left( 1 - \frac{v + (2l+1)\hbar}{u} \right) \right] A^+(u) A^-(v), \\
\{A^-(u), A^+(v)\} &= \hbar \frac{\partial}{\partial u} \log \left[ \prod_{l \geq 0} \left( 1 - \frac{u + (2l+1)\hbar}{v} \right) \right] A^-(u) A^+(v),
\end{aligned}$$

and further,

$$\begin{aligned}
\{\Lambda^+(u), \Lambda^-(v)\} &= \hbar \frac{\partial}{\partial v} \log \rho(u-v) \Lambda^+(u) \Lambda^-(v), \\
\{\Lambda^-(u), \Lambda^-(u)\} &= -\hbar \frac{\partial}{\partial u} \log \rho(v-u) \Lambda^-(u) \Lambda^+(v).
\end{aligned}$$

Remembering the definition (19) of  $\Lambda(u)$ , we have

$$\{\Lambda(u), \Lambda(v)\} = \hbar \left[ \frac{\partial}{\partial v} \log \rho(u-v) - \frac{\partial}{\partial u} \log \rho(v-u) \right] \Lambda(u) \Lambda(v).$$

Finally, we have for

$$l(u) \rightarrow s(u) = \Lambda(u - \frac{1}{2}\hbar) + \Lambda(u + \frac{1}{2}\hbar)^{-1}$$

the following Poisson bracket,

$$\begin{aligned}
\{s(u), s(v)\} &= \hbar \left[ \frac{\partial}{\partial v} \log \rho(u-v) - \frac{\partial}{\partial u} \log \rho(v-u) \right] s(u) s(v) \\
&\quad + \hbar \delta(u-v-\hbar) - \hbar \delta(u-v+\hbar),
\end{aligned}$$

where the  $\delta$ -functions are defined in the same way as in eq.(1).

Before concluding this paper let us consider the connections between the  $\hbar$ -deformed Miura map and some  $\hbar$ -difference equations. These are analogs of the well-known Gelfand-Dickey equations written in the simplest case: the classical Miura map

$$\partial^2 - q(t) = \left( \partial - \frac{1}{2}\chi(t) \right) \left( \partial + \frac{1}{2}\chi(t) \right). \quad (21)$$

Let  $\mathcal{D}_\hbar$  be the  $\hbar$ -“derivative” defined as

$$\mathcal{D}_\hbar Q(u) = Q(u - \hbar).$$

Then the  $\hbar$ -deformed version of eq.(21) can be written as

$$\left(\Lambda(u - \frac{\hbar}{2})\mathcal{D}_\hbar - 1\right)\left(\Lambda(u + \frac{\hbar}{2})^{-1}\mathcal{D}_\hbar - 1\right) = \mathcal{D}_\hbar^2 - s(u)\mathcal{D}_\hbar + 1.$$

In particular, if  $Q(u)$  is a solution of the  $\hbar$ -difference equation

$$(\mathcal{D}_\hbar^2 - s(u)\mathcal{D}_\hbar + 1)Q(u + \hbar) = 0,$$

then  $s(u)$  can be expressed in the form

$$s(u) = \frac{Q(u - \hbar)}{Q(u)} + \frac{Q(u + \hbar)}{Q(u)}.$$

This equation is completely in analogy to the  $q$ -deformed version given in Ref.[15], which was first used by Baxter [5] in studying eight vertex model. Such kind of equations have close connections to Bethe ansatz equations.

## 6 Concluding remarks and out-looking

In this paper we constructed the  $\hbar$ -deformed Miura map and the corresponding Virasoro algebra. This algebra can be viewed as either a Yangian deformation of the classical Virasoro algebra or a scaling limit of the  $q$ -deformed Virasoro algebra obtained by Frenkel and Reshetikhin in [15]. However, the construction in this paper is much more complicated than the  $q$ -deformed case because the  $\hbar$ -deformed Sugawara operator involves an infinite product structure in terms of the bosonic expression  $\Upsilon^-(u)$ . As a classical (Poisson bracket) algebra, our algebra is the classical limit of the quantum  $\hbar$ -deformed Virasoro algebra obtained from the quantum  $q$ -deformed Virasoro algebra by taking proper limit, and may also be viewed as a deformation of the conventional quantum Virasoro algebra governing the conformal field theory.

It is desirable that there exists an infinite family of  $\hbar$ -deformed algebras—call them  $\hbar$ -deformed  $W$ -algebras—each corresponds to a Yangian double of different underlying Lie algebras. The  $q$ -deformed  $W$ -algebras are already constructed in [15] and [28, 24, 2, 3, 13] both in classical and quantum form. However the  $\hbar$ -deformation of  $W$ -algebras is not known yet and we hope to work out this problem in our next publication.

Perhaps the most important application of the conventional classical  $W$ -algebras (i.e. Gelfand-Dickey Poisson algebras) is in the Hamiltonian description of integrable hierarchies such as the KdV hierarchy. For  $q$ -deformed  $W$ -algebras the corresponding differential-difference systems as deformed integrable hierarchies were obtained in [14]. It is interesting to perform the analogous constructions for  $\hbar$ -deformed  $W$ -algebras.

In the conventional (undeformed) cases,  $W$ -algebra generators are connected to principal co-minors of certain Wronskian determinants. It is quite interesting to ask the question that

whether the  $q$ - and/or  $\hbar$ -analogs exist. Such analogies are important if we want to identify the existence of the  $q$ - and/or  $\hbar$ -versions of nonstandard  $W$ -algebras, namely  $W$ -algebras beyond the standard  $W_n$  series,  $W_3^{(2)}$  for instance. We also hope to consider these problems in our future studies.

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